PERGAMON

International Journal of Solids and Structures 37 (2000) 6933–6960

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On the physical significance of higher order kinematic and static variables in a three-dimensional shell formulation

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Received 10 November 1999

Abstract

In recent years, considerable attention has been given to the development of higher order plate and shell models. These models are able to approximately represent three-dimensional effects, while pertaining the efficiency of a two-dimensional formulation due to pre-integration of the structural stiffness matrix across the thickness. Especially, the possibility to use unmodified, complete three-dimensional material laws within shell analysis has been a major motivation for the development of such models.

While the theoretical and numerical formulation of so-called 7-parameter shell models, including a thickness stretch of the shell, has been discussed in numerous papers, no thorough investigation of the physical significance of the additional kinematic and static variables, coming along with the extension into three dimensions, is known to the authors. However, realization of the mechanical meaning of these quantities is decisive for both a proper modeling of shell structures, e.g. concerning loading and kinematic boundary conditions, and a correct interpretation of the results. In the present paper, the significance of kinematic and static variables, appearing in a 7-parameter model proposed by Büchter and Ramm (1992a) are discussed. It is shown, how these quantities 'refine' the model behavior and how they can be related to the 'classical' variables, such as 'curvatures' and 'stress resultants'.

Furthermore, the special role of the material law within such a formulation is addressed. It is pointed out that certain requirements must hold for the variation of kinematic and static variables across the thickness, to ensure correct results. In this context it is found, that the considered 7-parameter model can be regarded as 'optimal' with respect to the number of degrees of freedom involved. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Higher order shell theory; Three-dimensional material law; Stress-resultants

1. Introduction

1.1. Step back into three dimensions

The ingenious idea of our ancestors using thin-walled structures to project the three-dimensional behavior onto a two-dimensional surface was not only a considerable reduction of mathematical expense but

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PII: S0020-7683(99)00321-2

it also gave the necessary physical insight into the structural response; thus, important phases such as membrane state, bending edge effects or inextensional deformations became apparent. The corresponding analytical solutions have been intensively investigated, mostly for special cases such as spherical, conical or cylindrical shells, in general, based on the so-called Kirchhoff–Love assumption (Kirchhoff, 1850; Love, 1888) neglecting transverse shear deformations. Names such as Reissner, Meissner, Geckeler, Flügge and many others are related to these theories. Besides this, higher order theories have been developed, including extra effects, for example those of Reissner's (Reissner, 1944) and Mindlin's (Mindlin, 1951, see also Hencky, 1947) extensions including transverse shear effects. The valuable contributions of Koiter (see e.g. Koiter, 1960) are milestones on the way to geometrically non-linear shell formulations.

When the *finite element method* became successful in the sixties, it was obvious to apply simply three-dimensional 'brick' elements also to plates and shells; a linear displacement field across the thickness (one node on the upper and lower shell surface, respectively), representing Reissner–Mindlin kinematics, was the natural choice. The failure of this procedure was attributed to a deficient representation of the Poisson effect and the large differences in stiffness; only an increase in the number of degrees of freedom across the thickness could remove the defect. This, in turn, caused an increase in number of elements also in surface direction and a tremendous expense. Consequently, the finite element developers returned to shell formulations, like the Kirchhoff–Love theory or – because of the C^0-requirement – more frequently to shear deformation theories with 'Reissner–Mindlin' kinematics (Ahmad et al., 1970; Ramm, 1976; Simo et al., 1990; Krätzig, 1993, among others).

In this article different shell models are distinguished in view of the number of degrees of freedom for displacements (and strains) involved, see Fig. 1. Thus, the shell theory of the Kirchhoff–Love type, making use of the two in-plane displacements and the transverse deflection is called 3-parameter model. If transverse shear effects are taken into account (Reissner–Mindlin kinematics), two independent rotations have to be introduced, leading to a 5-parameter model. Since in Section 2 a two-dimensional beam formulation is described as a model problem, the corresponding designations are also given in Fig. 1.

It is known that these 'conventional' shell formulations are not sufficient when

1. large strain effects become dominant so that the thickness change has to be considered,
2. three-dimensional constitutive laws should be applied without any manipulation or reduction,
3. three-dimensional effects need to be studied, such as local stress concentrations and material failure, delamination, etc., or
4. the assumption of a straight director does not physically hold, as in laminates and composites.

	Two-dimensional beam model	General shell model	Based on complete material law?
Membrane and bending (Euler–Bernoulli / Kirchhoff–Love)	Two parameters	Three parameters	No
+ Transverse shear (Timoshenko / Reissner–Mindlin)	Three parameters	Five parameters	No
+ Thickness stretch	Four parameters	Six parameters	Yes (defective! Poisson thickness locking)
+ Linear thickness stretch	Five parameters	Seven parameters	Yes

Fig. 1. Different degrees of approximation for beam and shell models.

The obvious extension for the first three requirements is to include the thickness change as the sixth parameter, leading to a fully three-dimensional set of stresses and strains. However, the mechanical ingredients of the 6-parameter model are the same as mentioned for the three-dimensional continuum, rendering the same problems given above. The main reason is that the resulting linear distribution of the normal stress S^{33} in thickness direction is not balanced by the constant strain E_{33} evolving from this model. The 'parasitic' linear part of S^{33} is caused by the linear distributions of E_{11} and E_{22} in thickness direction due to the Poisson effect. Thus, Poisson thickness locking occurs in bending dominated cases when Poisson's ratio is not equal to zero.

As a remedy, either the linear stress term has to be removed from the constitutive law or the shell formulation has to be extended by a linear strain term, leading to a 7-parameter model. This extension can either be achieved directly within a multifield variational formulation, as proposed in Büchter and Ramm (1992a) (see also Büchter et al., 1994) and adopted later on by others (Betsch et al., 1996; Eberlein and Wriggers, 1997), or indirectly within a displacement formulation by a quadratic variation of the transverse displacement in thickness direction (Verhoeven, 1993; Parisch, 1993; Sansour, 1995; Basar and Ding, 1996). In both cases, fully three-dimensional constitutive models can be applied without any modification.

Requirement 4 can only be handled by a refinement of the displacement field in thickness direction. This can either be achieved in the sense of a '*p*-refinement', by using higher order polynomials (Naghdi, 1972; Babuska and Li, 1991; Lo et al., 1977; Schwab, 1996, among others), i.e. a 'curved' director, or an '*h*-refinement', leading to the so-called multidirector or multilayer models (Epstein and Huttelmaier, 1983; Reddy 1987; Braun et al., 1994, among others) with layerwise straight (or curved) directors.

From the efficiency point of view, all these formulations include the key feature of shell analysis, opposite to continuum models, namely the explicit integration of the three-dimensional stress state across the thickness, leading to the so-called stress resultants. However, the term 'stress resultant' may not be adequate in the case of higher order shell formulations like the 7-parameter model, as will be shown in the sequel.

1.2. Objective

The objective of this article is to investigate the physical significance of the kinematic and static variables appearing in the 7-parameter model presented by Büchter and Ramm (1992a). The aim is to provide more physical insight into the model behavior, which is necessary for a correct model of shell structures and the interpretation of numerical results. Here, for example the question of correct loading and a proper choice of boundary conditions are crucial.

In 3-parameter and 5-parameter models, kinematic quantities are membrane strains, curvatures and transverse shear strains. The energetically conjugate static quantities are the membrane forces, bending moments and transverse shear forces. Engineers are familiar with the physical significance of these quantities. Due to the additional degrees of freedom in the 7-parameter model, extra kinematic and static variables show up. This results on the one hand in the approximate representation of certain three-dimensional effects, which are neglected in classical shell formulations without thickness stretch. On the other hand, it turns out to be a sophisticated exercise to physically classify these higher order effects and to find clear, meaningful expressions for them. In particular, for the static variables, the interpretation as a resultant 'force' or 'moment' is not always possible.

Some basic features of these higher order strain and stress variables are explained in Section 2 for a two-dimensional beam introduced as a model problem. Sections 4 and 5 deal with the discussion of the physical significance of kinematic and static variables, as well as the corresponding boundary conditions, in the context of the three-dimensional shell formulation, respectively.

Different from a three-dimensional solid formulation for a 'resultant' shell theory, the thickness enters the constitutive law. Thus, the slenderness of the shell becomes an important parameter, governing the relations between membrane, bending and transverse shear stiffness. When unmodified, three-dimensional material laws are applied to shell formulations, it is essential, that kinematic and static variables are 'balanced', which can clearly be seen from the format of the material tensor. This topic is discussed in Section 6 in the context of the 7-parameter shell model.

1.3. Extension based on Hu–Washizu functional

It has been mentioned before, that the 6-parameter model – although it already represents the full three-dimensional stress and strain state – needs an extension in the thickness direction in order to obtain sensible results. Several authors (Parisch, 1993; Sansour, 1995) have realized this extension by adding a quadratic term to the transverse displacement field, thus increasing the total number of degrees of freedom per node in a finite element discretization to 7 compared to 5 in a conventional, shear deformable shell formulation.

Motivated by the need to obtain a more efficient formulation, Büchter and Ramm (1992a) proposed a different procedure, namely to supplement the transverse normal strain E_{33} by a linear component β_{33} in thickness direction (see Eqs. (10) and (13)). This is achieved with the help of the *enhanced assumed strain* (*EAS*) *method*, originally introduced by Simo and Rifai (1990) to remove locking from low order plane strain/stress finite elements. The additional strain component is related to a single parameter $\tilde{\beta} = (h/2)\beta_{33}$, describing a linear variation of the transverse normal stretch across the thickness. The method to introduce this additional kinematic variable is derived from a version of the three-field variational functional of Hu–Washizu, depending on displacements $\boldsymbol{u}$, Green–Lagrange strains $\boldsymbol{E}$ and second Kirchhoff–Piola stresses $\boldsymbol{S}$. Following the concept of Simo and Rifai (1990), the strains are reparametrized by

$$\boldsymbol{E} = \boldsymbol{E}^u + \tilde{\boldsymbol{E}}, \quad \tilde{\boldsymbol{E}} = \frac{h}{2}\theta^3\beta_{33}\boldsymbol{g}^3 \otimes \boldsymbol{g}^3, \tag{1}$$

where $\boldsymbol{E}^u$ are the 'compatible strains' obtained from the displacement field, $\tilde{\boldsymbol{E}}$ are additional, *enhanced strains*, for which independent trial functions can be chosen, $\boldsymbol{g}^3$ are the contravariant base vectors in the θ^3-direction. Provided that a certain orthogonality condition holds, the stresses drop out of the variational formulation and the remaining *Hu–Washizu functional* reads

$$\tilde{\Pi}(\boldsymbol{u},\tilde{\boldsymbol{E}}) = \int_V W^{\text{int}}(\boldsymbol{u},\tilde{\boldsymbol{E}})\,\mathrm{d}V - \int_V \varrho\boldsymbol{b}\cdot\boldsymbol{u}\,\mathrm{d}V - \int_{A_S} \hat{\boldsymbol{t}}\cdot\boldsymbol{u}\,\mathrm{d}A = \text{stat.} \tag{2}$$

Here $W^{\text{int}}(\boldsymbol{u}, \tilde{\boldsymbol{E}})$ is the energy density, $\varrho\boldsymbol{b}$ are the body forces and $\hat{\boldsymbol{t}}$ are the boundary tractions. V denotes the volume of the shell body, A_S the portion of its surface subjected to prescribed forces. In the functional (2), it has already been taken into account, that the shape functions for the displacement field satisfy the kinematic boundary conditions, which is necessary for variational consistency.

Apparently, the format of Eq. (2) resembles that of the principle of virtual work, except that $W^{\text{int}}(\boldsymbol{u}, \tilde{\boldsymbol{E}})$ depends not only on displacements $\boldsymbol{u}$, but also on enhanced strains $\tilde{\boldsymbol{E}}$. The additional strain parameters, resulting from its discretization can be condensed on the element level, thus the number of degrees of freedom is not increased on the structural level.

The EAS method is related to the method of incompatible modes (Taylor et al., 1976), which has already been outlined in the original contribution by Simo and Rifai (1990). In fact, the enhanced strains could be interpreted as being linked to an incompatible displacement field that does not satisfy the kinematic boundary conditions or the interelement continuity conditions, respectively. The approach of Büchter and Ramm (1992a) can thus be interpreted as the addition of an incompatible field for the transverse displacement, which is quadratic in thickness direction (see Section 3.1).

As the enhanced strains $\tilde{\boldsymbol{E}}$ represent the residuum of the kinematic equation, it is expected that they vanish with mesh refinement. However, in the case of the 7-parameter model, for which a transverse strain component β_{33} is added, its vanishing would require refinement in thickness direction, which is usually not carried out. Thus, the enhanced strains remain present even for arbitrary mesh density in surface direction. This does, however, not affect consistency of the method, because of the sound variational basis. The fact, that $\tilde{\boldsymbol{E}}$ does not tend to zero with mesh refinement can be justified, since a shell model is not expected to represent *exactly* the three-dimensional continuum solution.

2. Model problem: two-dimensional beam with thickness change

Some of the mechanical phenomena occurring in shells, can in principle already be observed in a simple, plane beam. This holds for the membrane and bending states with their corresponding stress resultants. Only the twisting of the shell, corresponding to the torsion of a beam, is not present in a two-dimensional beam model.

The two-dimensional beam can thus serve as a model problem to investigate the principal differences between a formulation with and without thickness change. The advantage is that the mathematical formulation is much simpler as in a shell formulation without losing the physical insight into the appearing terms. The statements will afterwards be transferred to shells.

2.1. Kinematic assumptions for two-dimensional beam

For the two-dimensional beam formulation in curvilinear coordinates θ^1 denotes the direction of the beam's center line and θ^3 (instead of θ^2, for an easier comparison to the shell equations) denotes the transverse direction (Fig. 2). Therefore, Latin indices take the values 1 and 3, the same is valid when applying Einstein's summation convention, e.g.

$$\boldsymbol{u} = u_i \boldsymbol{g}^i = u_1 \boldsymbol{g}^1 + u_3 \boldsymbol{g}^3, \tag{3}$$

where $\boldsymbol{g}^i$ are the contravariant base vectors.

The variational basis of the beam formulation is the Hu–Washizu principle Eq. (2).

A 5-parameter beam model (see Fig. 1 for classifications) is described through the following assumptions for geometry:

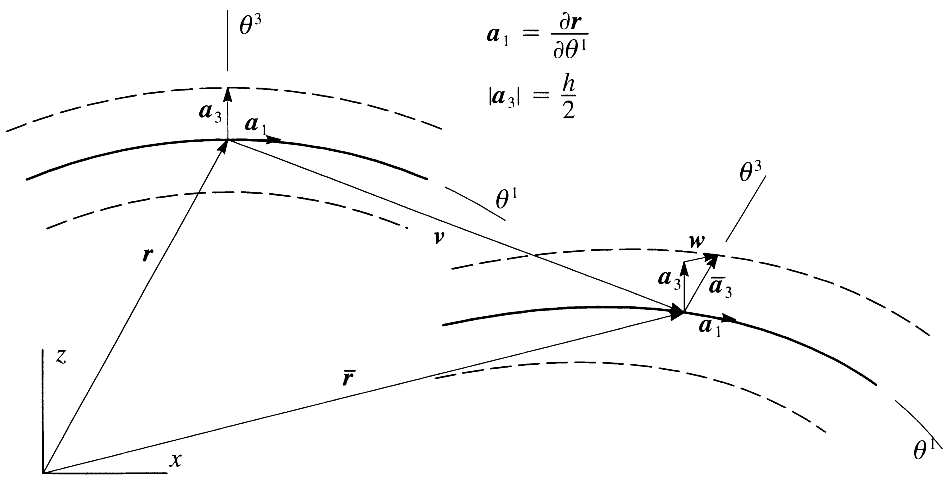


Fig. 2. Kinematics of two-dimensional beam.

$$\boldsymbol{x} = \boldsymbol{r} + \theta^3 \boldsymbol{a}_3, \tag{4}$$

displacements

$$\boldsymbol{u} = \boldsymbol{v} + \theta^3 \boldsymbol{w} \tag{5}$$

and strains

$$\beta_{33} = \frac{2}{h}\tilde{\beta} \text{ with } h = \text{thickness of beam.} \tag{6}$$

The degrees of freedom are the displacements $\boldsymbol{v}$, difference displacements $\boldsymbol{w}$, and the enhanced transverse normal strain component $\tilde{\beta}$. Note that difference displacements are introduced as primary variables. Similar to rotations, it keeps the formulation less prone to ill-conditioning if the structure is very thin.

The covariant base vectors of the beam's center line are given by

$$\boldsymbol{a}_1 = \frac{\partial \boldsymbol{r}}{\partial \theta^1} = \boldsymbol{r}_{,1} \tag{7}$$

and $\boldsymbol{a}_3$. The covariant base vectors outside the beam axis can be expressed in terms of $\boldsymbol{a}_i$.

$$\boldsymbol{g}_i = \boldsymbol{x}_{,i} \Rightarrow \begin{cases} \boldsymbol{g}_1 = \boldsymbol{a}_1 + \theta^3 \boldsymbol{a}_{3,1}, \\ \boldsymbol{g}_3 = \boldsymbol{a}_3. \end{cases} \tag{8}$$

The curvilinear components of the Green–Lagrange strain tensor for arbitrarily large displacements and strains $E_{ij} = E^u_{ij} + \tilde{E}_{ij}$ are made up of the displacement dependent strains

$$E^u_{ij} = \tfrac{1}{2}(\boldsymbol{g}_i \cdot \boldsymbol{u}_{,j} + \boldsymbol{g}_j \cdot \boldsymbol{u}_{,i} + \boldsymbol{u}_{,i} \cdot \boldsymbol{u}_{,j}) \tag{9}$$

and the enhanced strains

$$\tilde{E}_{ij} = 0 \text{ for } (i,j) \neq (3,3); \qquad \tilde{E}_{33} = \frac{h}{2}\theta^3 \beta_{33} = \theta^3 \tilde{\beta}. \tag{10}$$

The components of the strain tensor depending on the displacements can be written in the following form:

$$\begin{aligned} E^u_{11} &= (\boldsymbol{a}_1 + \theta^3 \boldsymbol{a}_{3,1}) \cdot (\boldsymbol{v}_{,1} + \theta^3 \boldsymbol{w}_{,1}) + \tfrac{1}{2}(\boldsymbol{v}_{,1} + \theta^3 \boldsymbol{w}_{,1}) \cdot (\boldsymbol{v}_{,1} + \theta^3 \boldsymbol{w}_{,1}) \\ &= \boldsymbol{a}_1 \cdot \boldsymbol{v}_1 + \tfrac{1}{2}\boldsymbol{v}_{,1} \cdot \boldsymbol{v}_{,1} + \theta^3[\boldsymbol{a}_1 \cdot \boldsymbol{w}_{,1} + \boldsymbol{a}_{3,1} \cdot \boldsymbol{v}_{,1} + \boldsymbol{v}_{,1} \cdot \boldsymbol{w}_{,1}] \\ &\quad + (\theta^3)^2\left[\boldsymbol{a}_{3,1} \cdot \boldsymbol{w}_{,1} + \tfrac{1}{2}\boldsymbol{w}_{,1} \cdot \boldsymbol{w}_{,1}\right], \end{aligned} \tag{11}$$

$$\begin{aligned} E^u_{13} &= \tfrac{1}{2}\left[(\boldsymbol{a}_1 + \theta^3 \boldsymbol{a}_{3,1}) \cdot \boldsymbol{w} + \boldsymbol{a}_3 \cdot (\boldsymbol{v}_{,1} + \theta^3 \boldsymbol{w}_{,1}) + (\boldsymbol{v}_{,1} + \theta^3 \boldsymbol{w}_{,1}) \cdot \boldsymbol{w}\right] \\ &= \tfrac{1}{2}[\boldsymbol{a}_1 \cdot \boldsymbol{w} + \boldsymbol{a}_3 \cdot \boldsymbol{v}_{,1} + \boldsymbol{v}_{,1} \cdot \boldsymbol{w}] + \tfrac{1}{2}\theta^3[\boldsymbol{a}_{3,1} \cdot \boldsymbol{w} + \boldsymbol{a}_3 \cdot \boldsymbol{w}_{,1} + \boldsymbol{w}_{,1} \cdot \boldsymbol{w}] = E_{31}, \end{aligned} \tag{12}$$

$$E^u_{33} = \boldsymbol{a}_3 \cdot \boldsymbol{w} + \tfrac{1}{2}\boldsymbol{w} \cdot \boldsymbol{w}. \tag{13}$$

For the transition from a two-dimensional to a one-dimensional description of the beam structure, kinematic and static quantities are defined. The kinematic variables are – as in the shell theory – the constant, linear and quadratic parts of the strain distribution in thickness direction.

$$E_{ij} = \alpha_{ij} + \frac{h}{2}\theta^3 \beta_{ij} + \frac{h^2}{4}(\theta^3)^2 \gamma_{ij}. \tag{14}$$

The static variables α_{ij}, β_{ij} and γ_{ij} are obtained as energetically conjugate quantities to the kinematic variables. Here, the transformation of an infinitesimal volume element

$$\hat{\mu} = \frac{(\boldsymbol{g}_1 \times \boldsymbol{g}_2) \cdot \boldsymbol{g}_3}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|} \tag{15}$$

is usually approximated by $\hat{\mu} \approx |\boldsymbol{g}_3| = h/2$ in the definition of the material law. This assumption is also introduced below in the shell formulation.

The resulting kinematic and static variables can be grouped into the 'usual' terms, also appearing in a classical, shear deformable 3-parameter beam formulation of Timoshenko type and additional, 'higher order' terms, resulting from the thickness stretch.

For simplicity, the following formulas are based on the assumption of an initially straight beam: Thus, a cumbersome notation and the introduction of Christoffel symbols can be avoided because covariant derivatives are identical to simple partial derivatives. The strain components then can be expressed in terms of the components of the displacements and it holds:

$$\boldsymbol{a}_i \cdot \boldsymbol{v} = v_i, \quad \boldsymbol{a}_i \cdot \boldsymbol{w} = w_i, \quad \boldsymbol{a}_{3,1} = \boldsymbol{0}. \tag{16}$$

It should be emphasized that the straight beam assumption is only introduced to keep the mathematical expressions as simple as possible and does not affect the general validity of the preceding equations. With the abbreviation $\partial(\cdot)/\partial\theta^1 = (\cdot)'$, the following expressions are obtained. Underlined terms are due to geometrically non-linear effects.

Usual kinematic and static variables

$$\alpha_{11} = v_1' + \underline{\frac{1}{2}[(v_1')^2 + (v_3')^2]}, \qquad n^{11} = \int_{-1}^{1} S^{11}\frac{h}{2}\mathrm{d}\theta^3, \tag{17}$$

$$\beta_{11} = \frac{2}{h}[w_1' + \underline{v_1'w_1' + v_3'w_3'}], \qquad m^{11} = \int_{-1}^{1} \theta^3 S^{11}\frac{h^2}{4}\mathrm{d}\theta^3, \tag{18}$$

$$\gamma_{11} = \underline{\frac{2}{h^2}((w_1')^2 + (w_3')^2)}, \qquad s^{11} = \int_{-1}^{1} (\theta^3)^2 S^{11}\frac{h^3}{8}\mathrm{d}\theta^3, \tag{19}$$

$$\alpha_{13} = \frac{1}{2}[w_1 + v_3' + \underline{v_1'w_1 + v_3'w_3}], \qquad n^{13} = \int_{-1}^{1} S^{13}\frac{h}{2}\mathrm{d}\theta^3 = n^{31}. \tag{20}$$

Additional kinematic and static variables due to two-dimensional extension

$$\beta_{13} = \frac{1}{h}(w_3' + \underline{w_1'w_1 + w_3'w_3}), \qquad m^{13} = \int_{-1}^{1} \theta^3 S^{13}\frac{h^2}{4}\mathrm{d}\theta^3, \tag{21}$$

$$\alpha_{33} = w_3 + \underline{\frac{1}{2}(w_1^2 + w_3^2)}, \qquad n^{33} = \int_{-1}^{1} S^{33}\frac{h}{2}\mathrm{d}\theta^3, \tag{22}$$

$$\beta_{33} = \frac{2}{h}\tilde{\beta}, \qquad m^{33} = \int_{-1}^{1} \theta^3 S^{33}\frac{h^2}{4}\mathrm{d}\theta^3. \tag{23}$$

The stress and strain terms quadratic in θ^3 are usually neglected because of their minor significance in most problems. However, in the analysis of relatively thick beams (or shells), for strong curvatures or in the presence of large strains together with bending deformations, these strains can become important (Büchter et al., 1994).

Neglecting γ_{11} and s^{11} the internal energy can be expressed as

$$\Pi^{\text{int}} = \int_\ell \frac{1}{2}\left[\alpha_{11}n^{11} + \beta_{11}m^{11} + 2(\alpha_{13}n^{13} + \beta_{13}m^{13}) + \alpha_{33}n^{33} + \beta_{33}m^{33}\right] \mathrm{d}\theta^1, \tag{24}$$

where symmetry of the shear strains has been used.

2.2. Interpretation of kinematic and static variables of two-dimensional beam

2.2.1. Geometrically linear terms

The physical significance of the kinematic quantities can easily be illustrated by the corresponding deformations of an infinitesimal portion of the beam. For simplicity, for the time being only, the linear contributions to the deformation are taken into account. The non-linear effects are discussed in the subsequent section (Fig. 3).

The constant part of the strains α_{11} parallel to the axis of the beam describes its longitudinal extension (membrane strains for shells), the linear part β_{11} its curvature. The corresponding stress resultants are the normal force ('membrane force') n^{11} and the bending moment m^{11}.

The transverse shear strain α_{13} is the last kinematic variable, that still appears in Timoshenko's beam theory. Already here an interesting remark can be made, which is usually not discussed when beam, plate or shell theories are derived.

Due to symmetry usually $\alpha_{13}n^{13}$ and $\alpha_{31}n^{31}$ are combined to $2\alpha_{13}n^{13} = \gamma n^{13}$ with the shear deformation γ.

$$\alpha_{13}n^{13} + \alpha_{31}n^{31} = (\alpha_{13} + \alpha_{31})n^{13} = \gamma n^{13}. \tag{25}$$

With the definition of n^{13} according to Eq. (20) as the integral over the shear strains, acting in transverse direction, in a cross-section orthogonal to the axis of the beam, the interpretation as a 'transverse shear force' suggests itself.

However, we could as well distinguish both parts and define

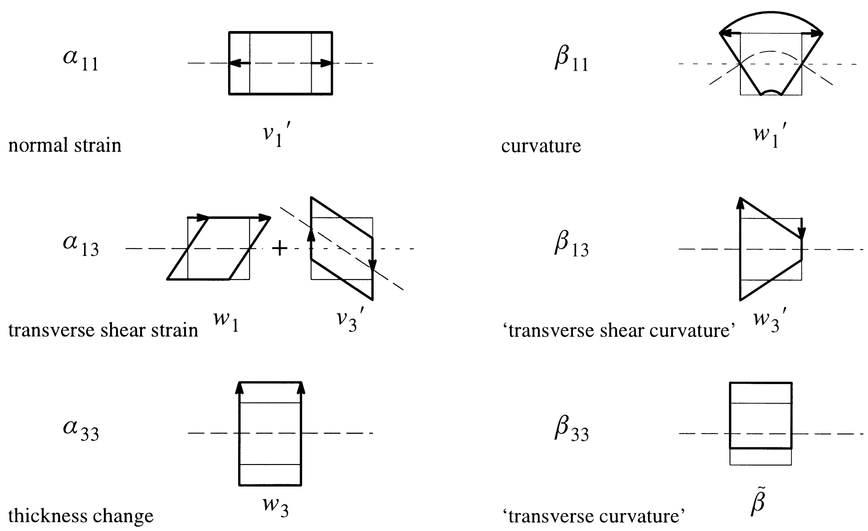


Fig. 3. Kinematic variables of two-dimensional beam.

$$n^{31} = \int_{-1}^{1} S^{31} \frac{h}{2} \mathrm{d}\theta^3 \tag{26}$$

to express the corresponding part of the internal energy. The stresses S^{31}, however, are not defined in the cross-section of the beam, but act in θ^1-direction on faces parallel to the beam's axis. The interpretation as a stress resultant is therefore not really correct for n^{31}, because it is not a resultant *force*, but merely an 'integral stress value'. Consequently, the transverse shear force is not the energetically conjugate quantity to the entire shear angle γ, but only to a part of it.

The question for the physical meaning of this quantity may seem to be rather academic, because the definition of γ and n^{13} suffices for the exact description of the internal energy. However, for the definition of the static variables described below, it is a necessary distinction, because they also do not fit into the usual framework of stress resultants and kinematics of a beam. These remaining quantities result from the assumption of a thickness change throughout deformation.

In Eq. (21), β_{13} and m^{13} are defined, which result from the linear distribution of the transverse shear strains. In contrast to α_{13}, β_{13} consists only of one part $((2/h)w_3')$, because $u_{1,3}$ is constant in θ^3-direction, whereas $u_{3,1}$ contains a linear part. Nevertheless, the same argument can be used as for the linear part of the transverse shear above. Also, here m^{31} could be defined independently as well.

Apart from this consideration, the question for the physical meaning of m^{13} remains. In Section 3.2, the term *transverse shear moment* is introduced for that quantity, inspired by the fact that the corresponding stresses vary linearly across the thickness. As for the bending moment m^{11}, the resultant force is zero, however, unlike m^{11}, the 'moment' m^{13} (and also m^{31}) does not lead to any resultant in the sense of a couple of forces. The transverse shear moment therefore, does not participate in the global equilibrium. Nevertheless, it should not be omitted because it contributes to the internal energy of the beam.

For the case of the *transverse normal force* n^{33}, the term 'stress resultant' is also not appropriate. In contrast to the transverse shear stresses, here no corresponding stresses exist in the cross-section of the beam. The name 'transverse normal force' takes into account that n^{33} is related to *normal stresses*, acting in *transverse* direction. The physical unit identifies n^{33} formally as a force, however, there is nothing like a resulting force in transverse direction. The mechanical interpretation of the corresponding kinematic quantity α_{33}, however, can easily be given as thickness change of the beam.

The corresponding linear part β_{33} is the only kinematic variable, that does not result from the displacements, but is introduced in this model as an independent degree of freedom within the enhanced assumed strain technique. It is related to a linear varying thickness change of the beam. The introduction of this term is necessary for an appropriate description of bending deformations together with non-vanishing Poisson's ratio ν. The energetically conjugate '*transverse moment*' m^{33} represents the linear distribution of transverse normal strains across the thickness, multiplied with the thickness coordinate (see Section 5.1).

2.2.2. Geometrically non-linear terms

Basically, the geometrically non-linear terms represent the effect of large strains and deformations within a certain 'deformation mode'. For example, in the case of α_{11}, the squares of the partial derivatives of the displacements v_1 and v_3 are added (see Eq. (17)). They represent the effects of finite strains and rotations for a fiber parallel to the θ^1-direction, as indicated in the 'one-dimensional truss analogy' in Fig. 4. Note that this effect is related to the choice of the Green–Lagrange tensor as strain measure. If instead Biot strains and stresses are used, the quadratic terms could be avoided. However, the choice of the strain measure is also closely related to the given constitutive law.

In principle, these non-linear parts do not affect the graphical descriptions of the physical significance of the corresponding kinematic variables, they are therefore, not further discussed in the sequel.

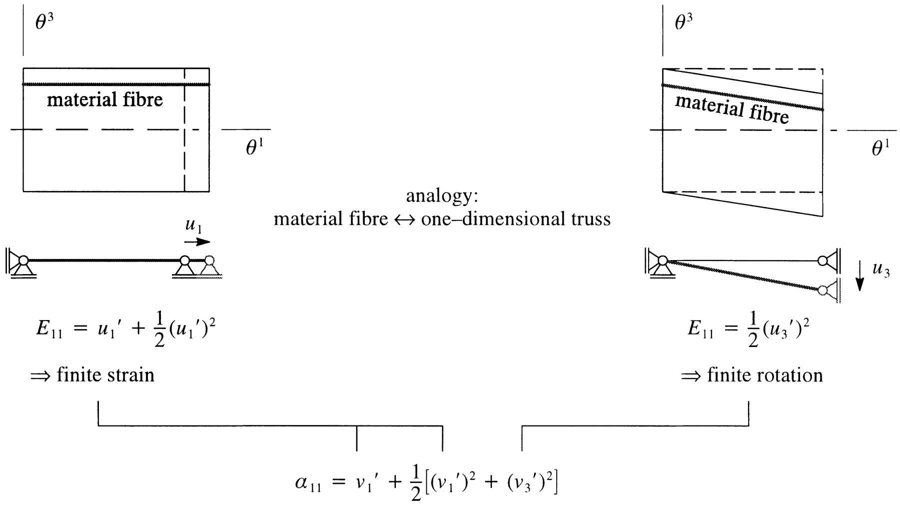


Fig. 4. Geometrically non-linear effects in the deformation of a single fiber.

However, the components γ_{11}, corresponding to the part of the strain tensor, which is quadratic in θ^3 (see Eq. (14)), are an exception, because no corresponding linear term exists. These components result from the linear distribution in θ^3 of the displacements.

In Fig. 5 it is illustrated that this quadratic variation of E_{11} consists of two parts. The first part $(w_1')^2$ accounts for the non-linear strain–displacement relation for the longitudinal stretching of a fiber parallel to the beam's center line. The indicated linear distribution of u_1 belongs to a bending deformation of the beam. Therefore, this part of γ_{11} is already present in a 3-parameter Timoshenko model. The second part $(w_3')^2$ only appears in a formulation that takes into account a thickness change of the beam throughout deformation. It considers the stretching of the same fiber in the case of a thickness change varying in θ^1-direction, which usually accompanies a 'membrane' type of deformation due to the Poisson effect. In a geometrically linear theory, the terms γ_{ij} vanish.

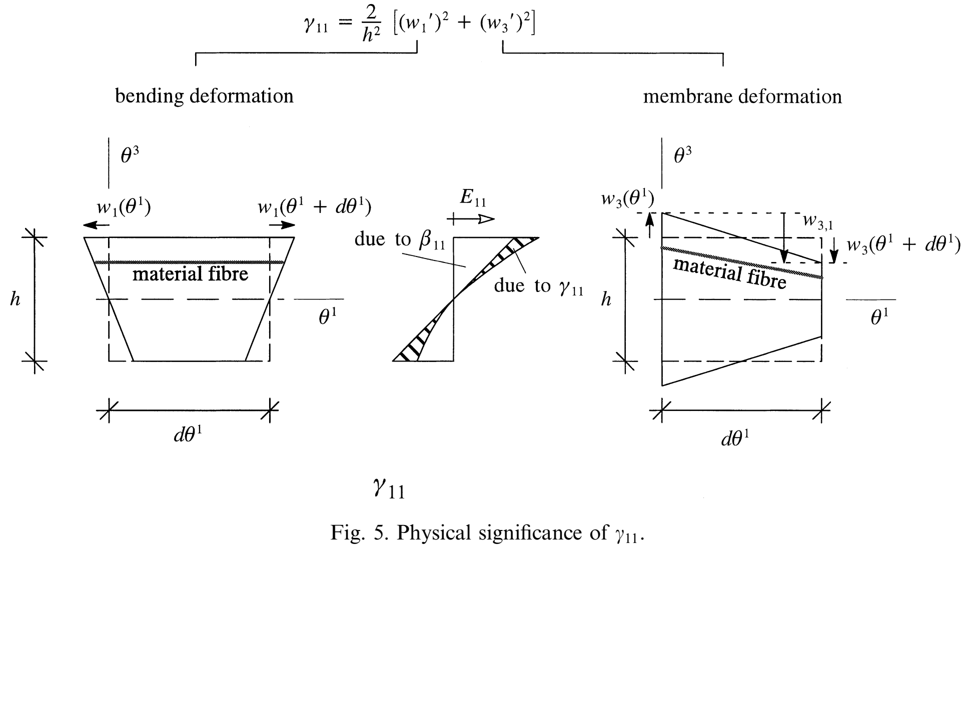


Fig. 5. Physical significance of γ_{11}.

2.3. Significance of constitutive law

As in the three-dimensional shell model, here also an unmodified, complete material law ought to be applied. In isotropic elasticity, following a St. Venant–Kirchhoff material law, this is the complete two-dimensional stress–strain relationship for plane stress conditions $\boldsymbol{S} = \boldsymbol{C} : \boldsymbol{E}$. It can be written in curvilinear components

$$S^{ij} = C^{ijkl} E_{kl} \tag{27}$$

or in matrix form

$$\begin{bmatrix} S^{11} \\ S^{33} \\ S^{13} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 \\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 \\ 0 & 0 & \frac{E}{2(1+v)} \end{bmatrix} \cdot \begin{bmatrix} E_{11} \\ E_{33} \\ 2E_{13} \end{bmatrix}. \tag{28}$$

The objective is to derive a relationship between kinematic and static variables without additional assumptions, like the zero normal stresses $S^{33} = 0$ in thickness direction. To this end, the material law (27) and the definition of the kinematic variables (14) are introduced into the definition for the integrated static variables, Eqs. (17–23).

$$n^{ij} = \int_{-1}^{1} S^{ij} \frac{h}{2} \mathrm{d}\theta^3 = \int_{-1}^{1} C^{ijkl} \frac{h}{2} \mathrm{d}\theta^3 \alpha_{kl} = D^{ijkl} \alpha_{kl}, \tag{29}$$

$$m^{ij} = \int_{-1}^{1} \theta^3 S^{ij} \frac{h^2}{4} \mathrm{d}\theta^3 = \int_{-1}^{1} (\theta^3)^2 C^{ijkl} \frac{h^3}{8} \mathrm{d}\theta^3 \beta_{kl} = \overline{D}^{ijkl} \beta_{kl}. \tag{30}$$

Thus, the components of the constitutive tensor $\boldsymbol{D}$ for the 5-parameter beam are obtained from pre-integration of $\boldsymbol{C}$ in thickness direction. For an initially flat beam it holds

$$D^{ijkl} = hC^{ijkl}, \qquad \overline{D}^{ijkl} = \frac{h^3}{12} C^{ijkl}. \tag{31}$$

From the matrix form, it can easily be seen, that the consequent derivation leads to a systematic and consistent format of $\boldsymbol{D}$.

$$\begin{bmatrix} n^{11} \\ n^{33} \\ n^{13} \\ m^{11} \\ m^{33} \\ m^{13} \end{bmatrix} = \begin{bmatrix} \frac{EA}{1-v^2} & \frac{vEA}{1-v^2} & 0 & 0 & 0 & 0 \\ \frac{vEA}{1-v^2} & \frac{EA}{1-v^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_q & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{EI}{1-v^2} & \frac{vEI}{1-v^2} & 0 \\ 0 & 0 & 0 & \frac{vEI}{1-v^2} & \frac{EI}{1-v^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & GI_q \end{bmatrix} \cdot \begin{bmatrix} \alpha_{11} \\ \alpha_{33} \\ 2\alpha_{13} \\ \beta_{11} \\ \beta_{33} \\ 2\beta_{13} \end{bmatrix}. \tag{32}$$

As typical for beams, here also in the two-dimensional formulation, the cross-sectional area A and the moment of inertia I have been introduced:

$$A = 1 \cdot h, \quad A_q = \alpha \cdot h, \quad I = \frac{1 \cdot h^3}{12}, \quad I_q = \beta \cdot \frac{h^3}{12}. \tag{33}$$

Here, the 'conventional' shear correction factor α, has been applied only for the constant part of the transverse shear strains α_{13} which is related to bending. The value of $\alpha = 5/6$ which is usually chosen can be derived from energetic considerations along with the quadratically varying shear strains across the thickness. However, the linear part of the shear strains can also be modified. A corresponding variation of the shear stresses should be antisymmetric (as the linear one) and satisfy the static boundary conditions. This reflection leads to the intuitive assumption of a cubic shear strain variation across the thickness. With the same considerations, used to obtain $\alpha = 5/6$, the resulting shear correction factor turns out to be $\beta = 7/10$. In Bischoff (1999), it is shown for shells, how the error with respect to the three-dimensional solution can be remarkably reduced with the help of this 'higher order' shear correction factor.

The constitutive law consists of two completely decoupled blocks of similar shape, reflecting also the format of the two-dimensional material law. Of course, they now depend on the thickness of the beam, namely on h in the constant part and on h^3 in the linear one. It can be seen that, here also, membrane and bending actions are decoupled at a material point for isotropic material laws. The vanishing terms within the blocks result from the fact that the θ^1- and θ^3-directions are orthogonal.

It is essential, that these blocks are complete. If, for example, $\tilde{\beta}$ would be omitted, an unbalance between kinematic and static variables emerges: While the linear part β_{33} of the kinematic variables drops out, due to Poisson's effect, m^{33} still shows up in the vector of static variables. This unbalance gives rise to non-physical, 'parasitic', normal stresses in thickness direction, the reason for 'Poisson thickness locking' with the consequence that pure continuum elements with a linear displacement field across the thickness do not work. The discussion of the requirement for completeness of the constitutive equations will be resumed in Section 6.

2.4. Equilibrium equations

Some interesting details, coming along with the extension in two dimensions of the beam formulation (or three dimensions for the shell theory), can be seen from the differential equations for static equilibrium. Only for simplicity and without loss of generality, geometric linearity is assumed here.

$$n^{11}_{,1} = -n^1, \tag{34}$$

$$n^{13}_{,1} = -n^3, \tag{35}$$

$$m^{11}_{,1} - \frac{h}{2} n^{13} = -m^1, \tag{36}$$

$$m^{13}_{,1} - \frac{h}{2} n^{33} = -m^3, \tag{37}$$

$$m^{33} = 0. \tag{38}$$

Here, n^i denotes a distributed load in θ^i-direction, m^1 stands for distributed moment loading. The load term m^3 can result from clamping effects, it causes stretching or compression of the beam in thickness direction (see Fig. 14).

Apparently, the first three equations are the usual equilibrium conditions, obtained from a Bernoulli or Timoshenko type of beam model. The three unknowns, n^{11}, n^{13} and m^{11} may be obtained from these equations alone, i.e. a beam theory incorporating only these quantities is statically determinate in the interior. Eqs. (37,38) are completely decoupled from these equations and involve only 'self equilibrated', higher order static variables. Since only one equation is available for two extra unknowns (n^{33} and m^{13}), they cannot be determined simply from static equilibrium. Here, compatibility conditions have to be taken into account, using material law and kinematic equations. Thus, the beam theory has become statically indeterminate in the interior.

Furthermore, it is interesting to note that the last Euler Eq. (38) requires that the 'transverse moment' m^{33} vanishes. This means, that the introduction of the extra strain parameter $\tilde{\beta}$ removes 'Poisson thickness locking' by explicitly setting the parasitic stresses to zero; in other words, the above mentioned unbalance between transverse normal stresses and strains is remedied. Here, two interesting remarks can be made, that also correspond to the three-dimensional shell.

Firstly, the fact that $m^{33} = 0$ in the continuous problem, regardless of loading and boundary conditions, seems somewhat surprising. The result can be understood by first realizing that the extra strain parameter $\tilde{\beta}$ does not appear in the kinematic boundary conditions of the underlying variational principle. It has been said before, that the corresponding strains can be regarded as resulting from an incompatible displacement field, which is not subject to any restriction, and consequently, forces cannot originate from a kinematic constraint. Yet, the extra strain parameter does show up in the static boundary conditions, thus a loading in this 'direction' is theoretically possible within the continuous variational formulation. In the finite element formulation, however, this is no longer given, because loads are applied on the structural level and $\tilde{\beta}$ is condensed on the element level.

Secondly, in a finite element formulation, Eq. (38) is satisfied only in a weak sense, which means that linearly varying transverse normal stresses appear for a finite number of elements and diminish with mesh refinement. It can, however, be observed that an in-plane discretization of $\tilde{\beta}$, which has the same order as the displacement field (e.g. quadratic distribution of $\tilde{\beta}$ in a three-noded beam element) satisfies Eq. (38) *exactly* for arbitrarily coarse meshes. This can be explained utilizing the analogy of the elements derived either from Hu–Washizu principle or from Hellinger–Reissner principle (see Andelfinger and Ramm, 1993; Bischoff et al., 1998). In fact, the above mentioned quadratic shape function for $\tilde{\beta}$ is equivalent to a Hellinger–Reissner based element, where the linear part of the transverse normal stresses is explicitly set to zero.

3. 7-parameter shell formulation including thickness stretch

3.1. Shell formulation using assumptions for displacements and strains

In Fig. 6, the reference and current configurations of a shell structure are shown along with the position vectors $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ of an arbitrary material point, the corresponding displacement vector $\boldsymbol{u}$ and the covariant base vectors $\boldsymbol{a}_i$ of the mid-surface of the shell. Basically, the notation is identical in both configurations, except that variables of the current configuration are denoted by a bar.

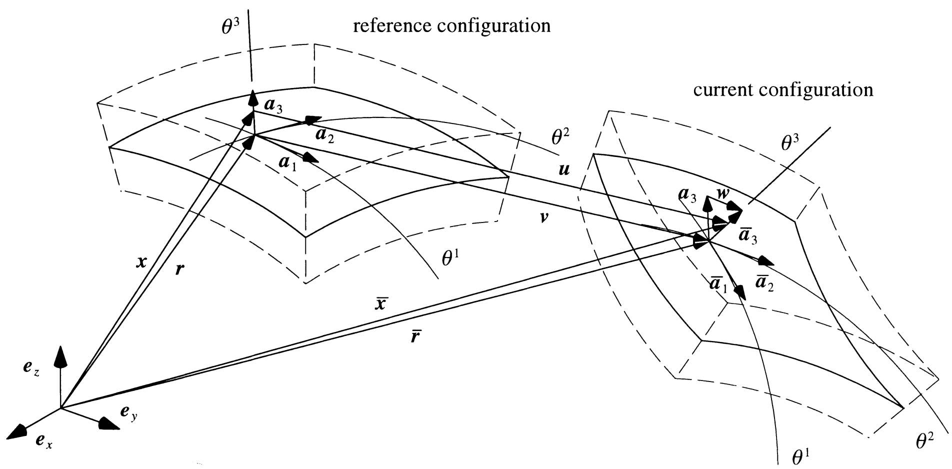


Fig. 6. Geometry and kinematics of 7-parameter shell model.

The description of the shell body is analogous to that of the above described beam formulation. The position vector $\boldsymbol{x}$ of an arbitrary point of the shell can be expressed with the help of $\boldsymbol{r}$ and $\boldsymbol{a}_3$.

$$\boldsymbol{x} = \boldsymbol{r} + \theta^3 \boldsymbol{a}_3, \qquad \overline{\boldsymbol{x}} = \overline{\boldsymbol{r}} + \theta^3 \overline{\boldsymbol{a}}_3. \tag{39}$$

The covariant base vectors of the mid-surface of the shell ($\theta^3 = 0$) are obtained from the partial derivatives of the position vector $\boldsymbol{r} = \boldsymbol{x}(\theta^3 = 0)$

$$\boldsymbol{a}_\alpha = \frac{\partial \boldsymbol{r}}{\partial \theta^\alpha} = \boldsymbol{r}_{,\alpha}, \qquad \overline{\boldsymbol{a}}_\alpha = \overline{\boldsymbol{r}}_{,\alpha}. \tag{40}$$

By definition, the director $\boldsymbol{a}_3$ is perpendicular to the mid-surface and has an initial length of $h/2$, where h is the shell thickness.

$$\boldsymbol{a}_3 = \frac{h}{2} \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}. \tag{41}$$

The covariant base vectors of an arbitrary point in the shell body are given by

$$\boldsymbol{g}_\alpha = \boldsymbol{x}_{,\alpha} = \boldsymbol{a}_\alpha + \theta^3 \boldsymbol{a}_{3,\alpha}, \qquad \boldsymbol{g}_3 = \boldsymbol{x}_{,3} = \boldsymbol{a}_3. \tag{42}$$

The displacement $\boldsymbol{v}$ of a point of the mid-surface together with the update of the director via the vector of the difference displacements $\boldsymbol{w}$

$$\overline{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{v}, \qquad \overline{\boldsymbol{a}}_3 = \boldsymbol{a}_3 + \boldsymbol{w} \tag{43}$$

renders an expression for the displacement $\boldsymbol{u}$ of an arbitrary point in the shell body.

$$\boldsymbol{u} = \overline{\boldsymbol{x}} - \boldsymbol{x} = \boldsymbol{r} + \boldsymbol{v} + \theta^3(\boldsymbol{a}_3 + \boldsymbol{w}) - \boldsymbol{r} - \theta^3 \boldsymbol{a}_3 = \boldsymbol{v} + \theta^3 \boldsymbol{w}. \tag{44}$$

Thus, the 7-parameter shell formulation utilizes first of all six degrees of freedom, evolving directly from the linear displacement assumption in thickness direction. These are three displacements of the mid-surface v_x, v_y and v_z, and three difference displacements w_x, w_y and w_z.

The pure displacement assumption is supplemented by a seventh degree of freedom, namely a linear distribution of transverse normal strains (see Eq. (52) for a definition of β_{ij})

$$\beta_{33} = \frac{2}{h} \tilde{\beta} \tag{45}$$

to avoid Poisson thickness locking. This extension is again realized with the help of the enhanced assumed strain technique based upon the Hu–Washizu principle, as described in Section 1.3. The additional, linear component of the transverse normal strain could alternatively be introduced via an incompatible, quadratic distribution of the transverse displacement in thickness direction. For details concerning the mathematical and numerical formulation of the 7-parameter shell formulation (Büchter and Ramm, 1992a; Bischoff and Ramm, 1997).

It is essential to realize that the extension to 7-parameters is generally necessary to ensure convergence to the correct solution, i.e. asymptotical correctness for the limit $h \to 0$. Although this extension is carried out with the help of a typical technique of 'finite element technology', it is thus rather an extension of the underlying *shell formulation* than one of *finite elements*.

3.2. Kinematic and static variables of the 7-parameter model

First of all, the 7-parameter model contains all strains and stress resultants appearing in a 5-parameter, Reissner–Mindlin type model. Nevertheless, they are also described herein, in order to allow for a relation to the additional kinematic and static variables and to introduce a clear notation. As in the beam formulation the displacements are assumed to vary linearly across the thickness (see Eq. (5)).

$$\boldsymbol{u} = \boldsymbol{v} + \theta^3 \boldsymbol{w}. \tag{46}$$

The components of the Green–Lagrange strain tensor consist of a part E^u_{ij} depending on the displacements and the enhanced part, depending on the extra parameter $\tilde{\beta}$.

$$E_{ij} = E^u_{ij} \quad \text{for} \ (i,j) \neq (3,3), \qquad E_{33} = E^u_{33} + \theta^3 \tilde{\beta}, \tag{47}$$

$$E^u_{ij} = \tfrac{1}{2}(\boldsymbol{g}_i \cdot \boldsymbol{u},_j + \boldsymbol{g}_j \cdot u,_i + \boldsymbol{u},_i \cdot \boldsymbol{u},_j). \tag{48}$$

With Eq. (42), the strain components depending on the displacements can be expressed in terms of quantities defined on the mid-surface of the shell.

$$\begin{aligned} E^u_{\alpha\beta} = \tfrac{1}{2}\big[&(\boldsymbol{a}_\alpha + \theta^3 \boldsymbol{a}_{3,\alpha}) \cdot (\boldsymbol{v},_\beta + \theta^3 \boldsymbol{w},_\beta) + (\boldsymbol{a}_\beta + \theta^3 \boldsymbol{a}_{3,\beta}) \cdot (\boldsymbol{v},_\alpha + \theta^3 \boldsymbol{w},_\alpha) \\ &+ (\boldsymbol{v},_\alpha + \theta^3 \boldsymbol{w},_\alpha) \cdot (\boldsymbol{v},_\beta + \theta^3 \boldsymbol{w},_\beta)\big], \end{aligned} \tag{49}$$

$$E^u_{\alpha 3} = \tfrac{1}{2}\big[(\boldsymbol{a}_\alpha + \theta^3 \boldsymbol{a}_{3,\alpha}) \cdot \boldsymbol{w} + \boldsymbol{a}_3 \cdot (\boldsymbol{v},_\alpha + \theta^3 \boldsymbol{w},_\alpha) + (\boldsymbol{v},_\alpha + \theta^3 \boldsymbol{w},_\alpha) \cdot \boldsymbol{w}\big], \tag{50}$$

$$E^u_{33} = \boldsymbol{a}_3 \cdot \boldsymbol{w} + \tfrac{1}{2}\boldsymbol{w} \cdot \boldsymbol{w}. \tag{51}$$

They can be decomposed into constant, linear and quadratic terms with respect to θ^3.

$$E_{ij} = \alpha_{ij} + \frac{h}{2}\theta^3 \beta_{ij} + \frac{h^2}{4}(\theta^3)^2 \gamma_{ij} \qquad \text{with } h = \text{thickness of shell}. \tag{52}$$

From Eqs. (46)–(48) we obtain kinematic variables of the 7-parameter shell model. For the same reasons as in Section 2.1 the assumption of an initially flat shell, i.e. a plate, is introduced also here. Thus,

$$\boldsymbol{a}_i \cdot \boldsymbol{v} = v_i, \quad \boldsymbol{a}_i \cdot \boldsymbol{w} = w_i, \quad a_{3,\alpha} = \boldsymbol{0}. \tag{53}$$

Again, this simplification is only carried out to avoid the otherwise necessary cumbersome notation and does not affect the validity of the 7-parameter model for general shells.

Following the above assumption, the strain components can be written as partial derivatives of the displacement components. Note that summation convention applies in the case of repeated indices.

$$\alpha_{11} = v_{1,1} + \tfrac{1}{2} v_{i,1} v_{i,1}, \tag{54}$$

$$\beta_{11} = \frac{2}{h}[w_{1,1} + v_{i,1} w_{i,1}], \tag{55}$$

$$\gamma_{11} = \frac{2}{h^2} w_{i,1} w_{i,1}, \tag{56}$$

$$\alpha_{12} = \tfrac{1}{2}[v_{1,2} + v_{2,1} + v_{i,1} v_{i,2}] = \alpha_{21}, \tag{57}$$

$$\beta_{12} = \frac{1}{h}[w_{1,2} + w_{2,1} + v_{i,1} w_{i,2} + v_{i,2} w_{i,1}] = \beta_{21}, \tag{58}$$

$$\gamma_{12} = 0 = \gamma_{21}, \tag{59}$$

$$\alpha_{22} = v_{2,2} + \tfrac{1}{2} v_{i,2} v_{i,2}, \tag{60}$$

$$\beta_{22} = \frac{2}{h}[w_{2,2} + v_{i,2}w_{i,2}], \tag{61}$$

$$\gamma_{22} = \frac{2}{h^2}[w_{i,2}w_{i,2}], \tag{62}$$

$$\alpha_{13} = \tfrac{1}{2}[w_1 + v_{3,1} + v_{i,1}w_i] = \alpha_{31}, \tag{63}$$

$$\beta_{13} = \frac{1}{h}[w_{3,1} + w_{i,1}w_i] = \beta_{31}, \tag{64}$$

$$\gamma_{13} = 0 = \gamma_{31}, \tag{65}$$

$$\alpha_{23} = \tfrac{1}{2}(w_2 + v_{3,2} + v_{i,2}w_i) = \alpha_{32}, \tag{66}$$

$$\beta_{23} = \frac{1}{h}(w_{3,2} + w_{i,2}w_i) = \beta_{32}, \tag{67}$$

$$\gamma_{23} = \gamma_{32} = 0, \tag{68}$$

$$\alpha_{33} = w_3 + \tfrac{1}{2}w_i w_i, \tag{69}$$

$$\beta_{33} = \frac{2}{h}\tilde{\beta}, \tag{70}$$

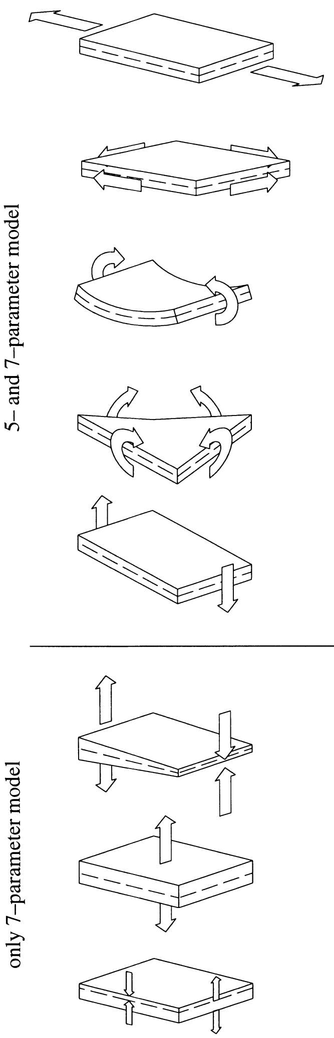
$$\gamma_{33} = 0. \tag{71}$$

The energetically conjugate static variables are defined as

$$n^{ij} = \int_{-1}^{1} S^{ij}\frac{h}{2}\mathrm{d}\theta^3, \qquad m^{ij} = \int_{-1}^{1} \theta^3 S^{ij}\frac{h^2}{4}\mathrm{d}\theta^3, \qquad s^{ij} = \int_{-1}^{1} (\theta^3)^2 S^{ij}\frac{h^3}{8}\mathrm{d}\theta^3. \tag{72}$$

In Fig. 7, all kinematic and static variables are listed schematically. For simplicity, the graphs show the deformation of a previously flat section of a ‘shell’. In addition to the information, which kinematic and static quantity is related to which deformation, it is indicated whether the deformation belongs to the membrane or the bending state of a shell.

Kinematic and static variables, are denoted in a strict scheme: Static quantities resulting from a constant distribution in θ^3 of normal stresses are called *normal forces*, those resulting from a linear distribution are *moments*. If the forces are based on shear stresses they are called *shear forces*. Consequently, the twisting moment should then be designated as *shear moment*, however, we stick to the established term *twisting moment*. If the underlying stresses are acting in thickness direction of the shell, the term ‘transverse’ is added. Thus, the constant distributions of the transverse shear stresses lead to *transverse shear forces*, the corresponding linear distributions yield the *transverse shear moment*. Finally, if the transverse normal stresses are integrated they lead to a *transverse normal force* and a *transverse moment*.



deformation	kinematic and static variables	denotation	membrane (M) or bending state (B)
	α_{11}, α_{22}	normal strains	(M)
	n^{11}, n^{22}	normal forces	
	α_{12}	shear strain	(M)
	n^{12}	shear force	
	β_{11}, β_{22}	curvature changes	(B)
	m^{11}, m^{22}	bending moments	
	β_{12}	twisting	(B)
	m^{12}	twisting moment	
	α_{13}, α_{23}	transverse shear strains	(B)
	n^{13}, n^{23}	transverse shear forces	
	β_{13}, β_{23}	transverse shear curvatures	(M)
	m^{13}, m^{23}	transverse shear moments	
	α_{33}	transverse normal strain	(M)
	n^{33}	transverse normal force	
	β_{33}	transverse curvature	(B)
	m^{33}	transverse moment	

Fig. 7. Deformations, kinematic and static variables of the shell.

4. Significance of kinematic quantities

4.1. Kinematic variables

All components α_{ij}, β_{ij} and γ_{ij}, of the Green–Lagrange strain tensor $\boldsymbol{E}$, arranged according to their distribution in θ^3-direction, are the kinematic variables. Due to their varying physical significance, it is sensible to distinguish between quantities according to

- their distribution in thickness direction,
- the direction of the corresponding stresses and
- the plane, in which they are acting.

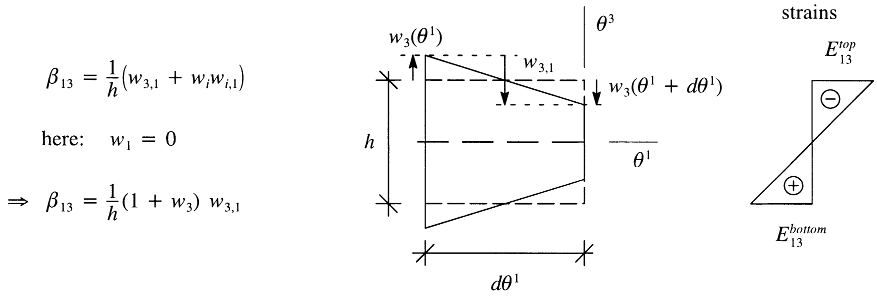


Fig. 8. Physical significance of $\beta_{\alpha 3}$.

It should be noted that the identification of strains with equal indices E_{ii} as 'normal strains' and those with mixed indices $E_{ij} \mid (i \neq j)$ as 'shear strains' is only valid, if θ^i and θ^j are orthogonal. The names used here rely on this supposition.

The constant parts of the strains parallel to the mid-surface of the shell $\alpha_{\alpha\beta}$ describe the *membrane strains*. Here, α_{11} and α_{22} are in-plane *normal strains* and α_{12} are in-plane *shear strains*. The linear components β_{11} and β_{22} are the *curvature changes*, corresponding to bending, β_{12} is called the *twisting* of the shell. The constant parts $\alpha_{\alpha 3}$ of the *transverse shear strains* are energetically conjugate to the transverse shear forces. In 5-parameter shell models, these are the only quantities, where stresses in θ^3-direction show up. In Kirchhoff–Love type shell theories these strains are constrained.

The linear portions $\beta_{\alpha 3}$ are only present in the 7-parameter shell model. From the graph of the corresponding deformation (Fig. 8), it can be seen that these strains follow from a thickness change of the shell.

Apart from a direct loading on the surface, for example, by clamping, these strains can result from membrane actions along with Poisson's effect. In regions with varying normal strains α_{11}, α_{22}, also varying thickness changes occur, leading to deformations shown in Fig. 8. In the overview in Fig. 7, the *transverse shear curvature* is therefore attributed to a membrane type of deformation.

The mechanical relevance of these strains for the load carrying behavior of shells is usually subordinate and might be neglected. In the present formulation, it is taken into account, because it is needed for a fully three-dimensional description of the strain state.

A quadratic component $\gamma_{\alpha 3}$ of the transverse shear strains cannot be described by the present type of a 7-parameter shell formulation, extended directly by a single strain parameter $\tilde{\beta}$. For pure displacement models based on a quadratic variation of the transverse displacements, however, such components show up. In this case, a shear correction factor is not necessary (Rössle et al., 1999).

The constant part of the normal strains in thickness direction α_{33} again only appears in the 7-parameter model. It is, however, implicitly present also in 3- and 5-parameter formulations as a dependent or secondary variable. In Mindlin's (Mindlin, 1951) and Kirchhoff's (Kirchhoff, 1850) theories, it can be calculated from the assumption of vanishing normal stresses in thickness direction. The contradiction to the assumption of an inextensible director ($\Rightarrow E_{33} = 0$) again illustrates the heuristic nature of these models. In Reissner's theory (Reissner, 1944), the normal strains are also eliminated from the formulation, however, there is no contradiction in this case, because Reissner started from a mixed functional, where the kinematic equation does not have to be fulfilled in a strong sense. In addition, Reissner did not use the assumption of vanishing stresses in thickness direction.

Like the linear part of the transverse shear strains, the transverse normal strains correspond to the membrane state of the shell. They could as well result from clamping, surface loading and Poisson's effect.

The linear part of the transverse normal strains β_{33} is again related to bending. As in the two-dimensional beam formulation, in the present 7-parameter model, this is the only strain component that does not

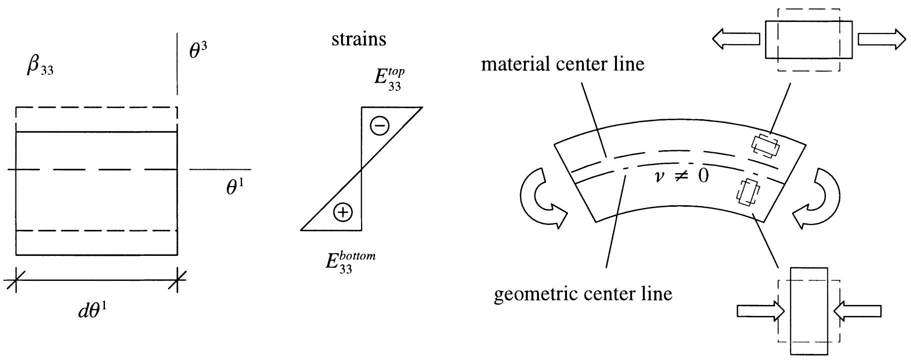


Fig. 9. Physical significance of β_{33}.

result from the displacement field but is introduced directly as extra degree of freedom within a multifield variational formulation.

The corresponding deformation of the shell can be described as a movement of its material center line within the shell body in transverse direction (see Fig. 9). This deformation usually occurs only in connection with bending; an external load which leads directly to this mode of deformation may be possible, but is not realistic.

For pure bending, as denoted in Fig. 9, the lower fibers are compressed in longitudinal direction and therefore, stretched in perpendicular direction due to the Poisson effect. Fibers on the upper side of the mid-plane show the opposite effect. In total the thickness of the shell remains unchanged, however, the mid-plane moves within the shell body to the side where it is stretched.

Of course, for a Poisson's ratio $\nu = 0$, this effect is not present. For such materials a 6-parameter formulation suffices. In general, omission of the *transverse curvature* β_{33} leads to artificial stiffening, called 'Poisson thickness locking' in the present paper.

4.2. Kinematic boundary conditions

Kinematic boundary conditions can only be prescribed for the six displacement components of the 7-parameter formulation. As already mentioned in Section 2.4, the seventh parameter is not subject to any kinematic constraints, due to the underlying variational formulation. However, in a 7-parameter formulation based on pure displacement assumptions, both a related load term and a kinematic boundary condition can be defined. The consequence is, that in the present version of a 7-parameter model normal strains in thickness direction can occur at a fully clamped edge of a shell. This could be interpreted as an incompatible, relative displacement of the mid-surface with respect to the upper and lower faces of the shell body. The same is valid, if the seventh parameter is introduced as an incompatible displacement, as it has been proposed by Parisch (1993).

The boundary conditions for the displacement degrees of freedom v_x, v_y and v_z have the same physical meaning as in classical plate and shell models. Suppressing the components w_x, w_y and w_z of the difference vector, however, has a slightly different effect compared to clamped boundary conditions constraining the rotational degrees of freedom. This is visualized in Fig. 10 for the special case of a plate with its mid-surface in the xy plane. Clamping the edge parallel to the y-axis is realized in a 7-parameter formulation suppressing the x-component $w_x = 0$ of the difference vector. In addition, a rotation around the x-axis may be constrained by $w_y = 0$ ('hard support'). The drilling rotation around the z-axis, which is offered in certain

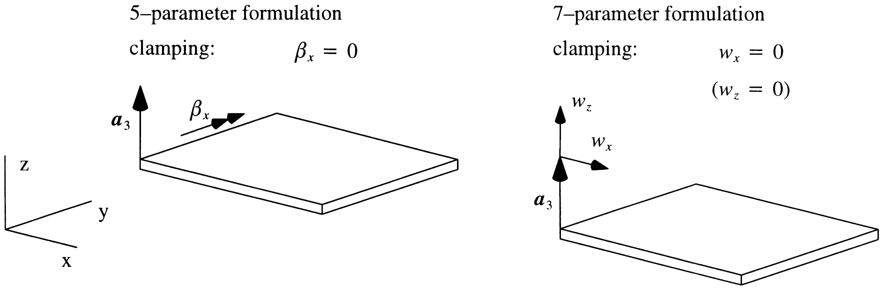


Fig. 10. Geometric boundary conditions.

shell formulations – especially in commercial FE packages to facilitate easy combinations with beam elements or discretization of shells with intersections – is not reproduced in the present formulation. Actually, such drilling degrees of freedom are often introduced artificially with some extra drilling stiffness or additional variables depending on the neighboring displacement field. There are also variationally sound derivations of membrane and shell formulations using drilling degrees of freedom existing in the literature. However, here also regularization parameters are needed, which can not always be identified uniquely. For a discussion of this topic see for example Ibrahimbegovic and Frey (1995).

The remaining sixth degree of freedom w_z of the 7-parameter model describes the thickness change of the shell. Thus, the choice has to be made, whether or not to suppress a thickness change, to ensure a proper representation of the actual physical situation. For example, w_z might be suppressed at the edge of a concrete slab, monolithically connected to a wall, but it has to be left free at edges, that are clamped due to symmetry conditions. The sixth degree of freedom also facilitates plane-strain calculations. In particular, it is possible to apply this condition selectively in certain regions within the structure. Thus, it is possible to approximately consider for example the effect of clamping tools in testing machines, which is impossible with 3- or 5-parameter shell models.

By combination of boundary conditions for displacements and difference displacements, it is, in principle, possible to model eccentric suspensions. For example, the condition

$$v_z - w_z = 0 \tag{73}$$

describes a support on the lower surface of the shell.

$$u_z = v_z + \theta^3 w_z = v_z + \theta^3 v_z \Rightarrow u_z = 0 \qquad \text{for } \theta^3 = -1. \tag{74}$$

This approximate description of three-dimensional effects is not possible when using shell models without thickness change.

In the analysis of shells with intersections, it is often referred to the problem that two different normals and thus two directors exist at each point along the intersection. For shell formulations using a rotation tensor, this leads to the problem of non-communicating rotational degrees of freedom and can, for example, be remedied by introducing a drilling rotation along the director as sixth parameter. For 'degenerated' shell models, either with five or seven parameters, it is natural to use a common director at these locations, as it would be the case for a corresponding mesh with three-dimensional 'brick' elements (Ahmad et al., 1970). Thus, the problem of non-communicating degrees of freedom is circumvented in a simple and elegant way. For a discussion of the error resulting from the fact that the director field is not normal to the shell surface (Büchter and Ramm, 1992b). In the case of sharp intersections, it can be verified at least numerically that this approach produces similar or even better results than the methods with coupling of rotational degrees of freedom (Bischoff, 1999).

5. Significance of static quantities

5.1. Static variables

The original idea of 'stress resultants' in the development of beam, plate and shell theories was presumably not the formal process of pre-integration across the thickness, but the reflection that the forces and moments in a cross section are directly related to equilibrium in a global sense. Yet, here the static variables n^{ij}, m^{ij} and s^{ij} are derived by formal thickness integration, so that they are energetically conjugated to the kinematic variables, defined in Section 4.1. As already mentioned, not all of these 'integrated stresses' can be interpreted as forces or moments because the corresponding stresses do not always act in the cross-section of the shell.

Integration of constant and linear components of $S^{\alpha\beta}$ leads to *membrane forces* $n^{\alpha\beta}$ and *moments* $m^{\alpha\beta}$, respectively. The components $s^{\alpha\beta}$, resulting from quadratic variation of the stresses $S^{\alpha\beta}$, are energetically conjugate to the previously described kinematic variables $\gamma_{\alpha\beta}$ (see Section 2.2.2). Physically, these static variables can be interpreted as *bi-moments* (Fig. 11). Here, an analogy may be observed to a related effect, occurring in warping torsion of thin-walled beams, where bi-moments occur due to warping of the cross-section of the beam, however, not related to the geometric non-linearity.

Describing the related kinematic variables $\gamma_{\alpha\beta}$ it has been stated, that these components are usually neglected, due to their minor influence on the structural behavior. The same is valid for the static variables $s^{\alpha\beta}$. For both $\gamma_{\alpha\beta}$ and $s^{\alpha\beta}$ it can be said, that they are not mandatory for the desired completeness of the three-dimensional formulation. The static variables $s^{\alpha\beta}$ are the only components in the present version, which vanish within a geometrically linear formulation.

The integrals over the constant parts of the transverse shear stresses $S^{\alpha 3}$ represent the *transverse shear forces* $n^{\alpha 3}$. Within a 5-parameter formulation, these are the only stresses acting in θ^3-direction. Although in a 7-parameter formulation additional static variables appear, where the corresponding stresses act in thickness direction, the transverse shear forces remain as the only resulting *forces* in thickness direction in a *cross-section* of the shell. This is due to the fact that the resultant forces of the remaining static variables (e.g. $m^{\alpha 3}$) either vanish, which means that they are self-equilibrated, or do not act in a cross-section (as e.g. n^{33}).

The integrals of the linear parts of the transverse shear stresses have been called *transverse shear moments* $m^{\alpha 3}$ in Fig. 7. They only appear in the 7-parameter formulation. The term 'moment' has been chosen, because the related stress distribution is linear in θ^3, as in the case of a bending or twisting moment. As in the two-dimensional beam the stresses, corresponding to $m^{\alpha 3}$, are 'self-equilibrated', the resulting 'moment' in the sense of a couple of forces, is zero (Fig. 12). If they arise from geometric or static boundary conditions, they only have a local influence on the structural behavior, because of St. Venant's principle. The transverse shear moments do not contribute to the satisfaction of global equilibrium.

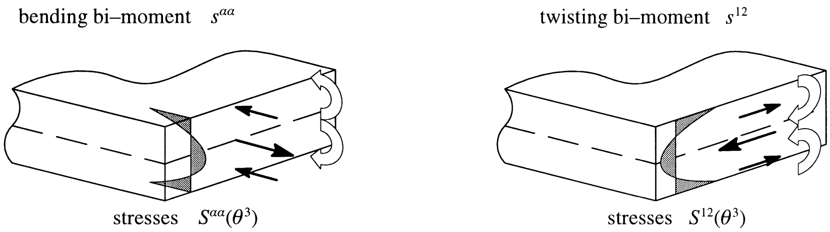


Fig. 11. Physical significance of $s^{\alpha\beta}$.

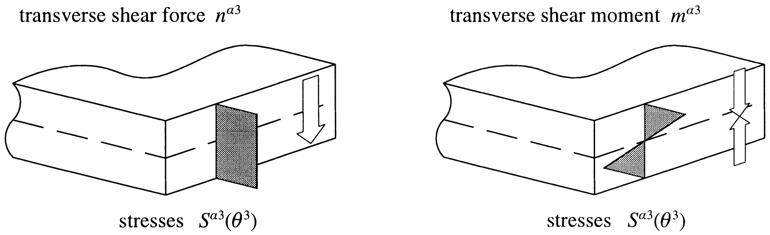


Fig. 12. Physical significance of $n^{\alpha 3}$ and $m^{\alpha 3}$.

The quadratic components of the transverse shear stresses are again zero ($s^{\alpha 3} = 0$), as in the two-dimensional beam formulation.

This ends the discussion of stresses acting in a cross-section. The remaining static variables correspond to transverse normal stresses S^{33}. As already mentioned in the discussion of the two-dimensional beam model problem, these are neither real forces nor do they show up in a section perpendicular to the shell. Nevertheless, the consideration of these integrated stresses is essential, because they contribute to the internal energy of the shell structure and represent the effect of transverse normal stresses in the shell.

The constant component in θ^3, the *transverse normal force* n^{33}, is related to normal stresses that evolve when the shell is stretched or compressed in thickness direction.

$$n^{33} = S^{33} h, \tag{75}$$

where h is the thickness of the shell. The unit (kN/m) of n^{33} is derived from (kN/m^2) (stress S^{33}) and (m) (thickness h). A simple physical interpretation of such a variable is not easy to find. If we intellectually reduce the three-dimensional problem to a one-dimensional model in thickness direction, n^{33} is identical to the force P in a truss, multiplied with it's length ℓ. Given an elongation $\Delta\ell$ of the truss, we have

$$\Delta\ell = \frac{P\ell}{EA} \iff P\ell = EA\Delta\ell = EA\int_0^\ell \varepsilon\,\mathrm{d}x = \int_0^\ell P\mathrm{d}x =: n^{33}. \tag{76}$$

Obviously, n^{33} relates to P (or S^{33}, in general), like $\Delta\ell$ to ε (or E^{33}, respectively). Thus, the transverse normal force might be designated as an 'amount of stress' in the structure, the value n^{33} results from 'squeezing' the stresses along h onto one single point (Fig. 13).

It is interesting to remark that the energetically conjugate strain α_{33} is, in contrast to that, easily accessible for a mechanical interpretation. It corresponds, most simply, to the thickness stretch of the shell at a certain point.

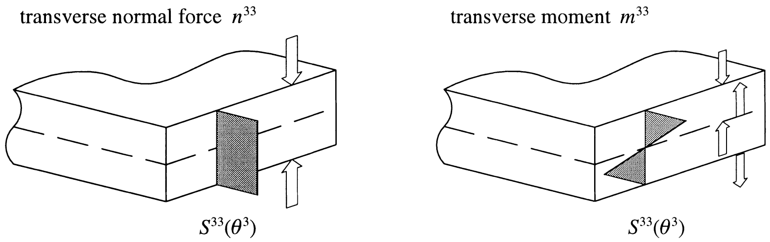


Fig. 13. Physical significance of n^{33} and m^{33}.

For the *transverse moment* m^{33} in principle the same could be said. Also, here the term 'moment' can only be justified by the systematic scheme introduced at the beginning of Section 3.2. A physical interpretation of m^{33} is probably indirectly possible with the help of the corresponding deformation. While the transverse normal force tends to stretch or compress the shell entirely in thickness direction, the transverse moment tries to elongate the upper part while it shortens the lower, or vice versa.

5.2. Static boundary conditions and loading

In contrast to what has been said in Section 4.2 for the geometric boundary conditions, in the case of the static boundary conditions, it is possible to prescribe values for the stress resultants, corresponding to the enhanced strains. In a finite element formulation, however, loads are defined on the structural level, i.e. after assembly of the global stiffness matrix. Since the enhanced strain parameters are eliminated on the element level, their related load terms are no longer present on the structural level.

Since static boundary conditions and loading in the domain have the same effect in a finite element analysis, only the domain terms are discussed in the following.

Most loads on shell structures – apart from volume loads – result from contact with other media on its top or bottom faces. In a two-dimensional model of shells, these loads are assumed to act upon its mid-surface, whereas in a 7-parameter formulation a more realistic load application can be modeled (Ramm et al., 1995). To illustrate this possibility, the domain term of the load is decomposed into components.

$$\int_V \boldsymbol{b} \cdot \delta \boldsymbol{u} \, \mathrm{d}V = \int_A \left[n^\alpha \delta v_\alpha + n^3 \delta v_3 + m^\alpha \delta w_\alpha + m^3 \delta w_3 \right] \mathrm{d}A. \tag{77}$$

The components acting in-plane n^α do not differ from that in classical shell formulations, the same is true for the transverse loading n^3 on the mid-surface of the shell. A distributed moment load m^α is mostly not taken into account, it can, however, be considered for 3- and 5-parameter shell models. It could possibly evolve from eccentric tangential forces, for example from exterior facades. Whereas, in a 5-parameter formulation, this moment is directly related to the rotational degrees of freedom, in the 7-parameter model; it corresponds to a component of the difference vector, parallel to the mid-surface of the shell. The resulting moment load follows as a couple of forces and depends on the shell thickness (see Fig. 14).

Finally, a fourth term m^3 may appear, consisting of two opposite, self-equilibrated loads, acting perpendicular to the shell surfaces. A combination of n^3 and m^3 can be used to model transverse loads acting upon the upper or lower surface of the shell (Ramm et al., 1995). Although this model leads to a slight improvement, it is still a rough approximation of the load application, which cannot serve as a substitute for a detailed analysis of the three-dimensional stress and strain state.

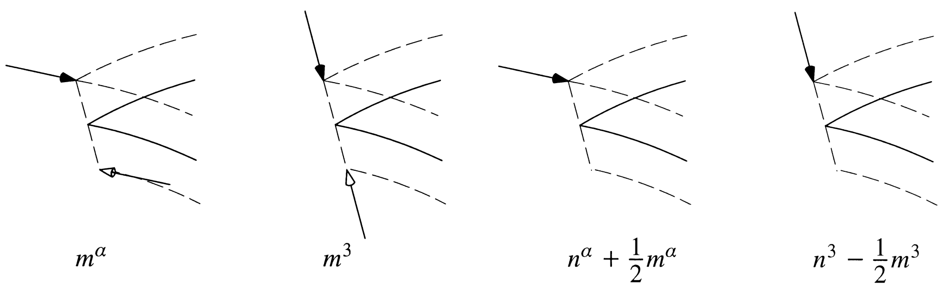


Fig. 14. Loading of difference vector components – surface loading.

In Fig. 14, the hollow arrow for the load terms m^α and m^3 on the lower side of the shell illustrates the fact, that loading of a component of the difference vector always results in a couple of forces in opposite directions. In a simulation of surface loads, this fact may be exploited to produce an 'eccentricity' of n^α or n^3.

6. Significance of material law

It has been mentioned previously that the use of complete three-dimensional material laws is one of the basic motivations for the development of the present shell model (Büchter and Ramm, 1992a). The definition of the material tensor does not require any assumption or modification for the transition from three to two dimensions. The unmodified, three-dimensional material tensor is simply integrated in thickness direction to obtain a constitutive relation between 'integrated' kinematic and static variables (replacing that of strains and stresses).

In a finite element formulation this 'pre-integration' is usually done numerically, in particular for heterogeneous materials or materially non-linear responses. In order to be able to discuss the mechanical significance of the single components, in this Section the components of a material tensor for a linear elastic, isotropic, homogeneous material law is derived explicitly. The derived conclusions can be generalized in the sense that for other materials values in the matrix may change and additional coupling terms can evolve, but their physical meaning basically remains.

The derivation of the 'integrated' constitutive relation is, in principle, the same as the one used for the two-dimensional beam. Starting point is the complete three-dimensional stress–strain relation for a linear elastic material law for small strains. As in the previous Sections, terms like 'shear' or 'tension' rely on the presumption, that θ^1- and θ^2-directions be orthogonal.

$$S^{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} \tag{78}$$

with the Lamé constants

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \qquad \mu = G = \frac{E}{2(1+v)}. \tag{79}$$

The material law can be cast in the following matrix form.

$$\begin{bmatrix} S^{11} \\ S^{22} \\ S^{33} \\ S^{12} \\ S^{13} \\ S^{23} \end{bmatrix} = \frac{\lambda}{v} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \cdot \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{13} \\ 2E_{23} \end{bmatrix}. \tag{80}$$

Its structure – as in the two-dimensional beam – will be reproduced in the constitutive law for the shell. The underlying definitions of the components of the three-dimensional material tensor

$$C^{ijkl} = \frac{Ev}{(1+v)(1-2v)} g^{ij} g^{kl} + \frac{E}{2(1+v)} \left[g^{ik} g^{jl} + g^{il} g^{kj} \right] \tag{81}$$

simplifies with the assumption of an orthogonal basis (i.e. a flat shell), because then $g^{ij} = \delta^{ij}$. For pre-integration across the thickness, the terms correlated with γ_{ij} (Eq. (52)) and s^{ij} (Eq. (72)) are omitted for simplicity (see Section 2.2.2); beside this the assumption $\hat{\mu} = h/2$ is adopted (see Section 2.1). The components of the constitutive tensor are then

$$D_K^{ijkl} = \int_{-1}^{1} (\theta^3)^K C^{ijkl} \left(\frac{h}{2}\right)^{(K+1)} \mathrm{d}\theta^3, \qquad K \in \{0, 1, 2\}. \tag{82}$$

With the abbreviations (α and β are the shear correction factors)

$$\overline{E} = \frac{E(1-v)}{(1+v)(1-2v)}, \qquad h_q = \alpha \cdot h, \ \overline{h} = \frac{h^3}{12}, \ \overline{h}_q = \beta \cdot \overline{h}, \tag{83}$$

the material law of the 7-parameter shell formulation gets the following form:

$$\begin{bmatrix} n^{11} \\ n^{22} \\ n^{33} \\ n^{12} \\ n^{13} \\ n^{23} \\ m^{11} \\ m^{22} \\ m^{33} \\ m^{12} \\ m^{13} \\ m^{23} \end{bmatrix} = \begin{bmatrix} \overline{E}h & \lambda h & \lambda h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda h & \overline{E}h & \lambda h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda h & \lambda h & \overline{E}h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Gh & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Gh_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Gh_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{Eh} & \lambda\overline{h} & \lambda\overline{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda\overline{h} & \overline{Eh} & \lambda\overline{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda\overline{h} & \lambda\overline{h} & \overline{Eh} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G\overline{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G\overline{h}_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G\overline{h}_q \end{bmatrix} \cdot \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{12} \\ 2\alpha_{13} \\ 2\alpha_{23} \\ \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 2\beta_{12} \\ 2\beta_{13} \\ 2\beta_{23} \end{bmatrix}. \tag{84}$$

As in the above described two-dimensional beam formulation with thickness change, the format of the original material law is reflected in Eq. (84). It consists of two completely decoupled blocks that only differ in the contribution of the shell thickness h, namely h for the constant stress components and $\overline{h} = h^3/12$ for the linear ones.

The introduction of the shear correction factors α and β may seem a contradiction to the use of *un-modified* constitutive laws. This is, however, not the case. Omission of α and β would not remove the asymptotic correctness of the shell formulation (i.e. convergence towards the three-dimensional solution for $h \to 0$), but only reduce accuracy for finite thickness.

The decisive fact in Eq. (84) is that there is a certain 'balance' between kinematic and static values. Each of the kinematic variables has its counterpart in the vector of static variables. Again, when $\tilde{\beta}$ is omitted the linear part of the transverse normal strains vanishes, while the equivalent static variable m^{33} is still present. This unbalance results in parasitic stresses in thickness direction leading to a deficient shell formulation. Since this unbalance between static and kinematic variables is the typical characteristic of locking phenomena this is also called 'Poisson thickness locking'.

6.1. 'Completeness' of shell formulations using three-dimensional material laws

We have learned that the 7-parameter shell formulation takes into account exactly the constant and linear variations in θ^3 of all strain and stress components. For isotropic material laws additional terms, like a quadratic distribution of transverse shear strains might refine the model behavior and improve accuracy of the results. In contrast to that, for general anisotropic material laws, it is not sensible to supplement the formulation by only particular components of the full set of strains and stresses, rather than the full set. The overall coupling of all stress and strain terms of the same order due to the general anisotropy could result in parasitic stresses and thus in an artificial constraint. Anyway, for a shell theory, which ought to describe only membrane and bending effects, such an extension is not necessary. To sum up, one can say that, having in mind the requirement to be able to apply arbitrary constitutive laws, only a *complete* linear, quadratic, cubic, etc. distribution of the stress–strain state seems to be sensible.

At the same time, this means that the 7-parameter formulation, in the special form proposed by Büchter and Ramm (1992a) is the version of a three-dimensional, shear deformable shell theory with the lowest possible effort regarding the number of degrees of freedom involved.

Libai and Simmonds (1998) support the view that the Kirchhoff theory needs to be derived from *constitutive* assumptions alone and not from a priori *kinematic* assumptions, which is usually done. With the help of a certain construction of the constitutive law of the shell, it is possible to eliminate the terms, that should not produce energy (for example transverse normal strains and stresses). This point of view is closely related to the 'plane stress' assumption used by Koiter (1960) in the sense that only stresses acting within a certain plane are taken into account, while defining the internal energy. Thus, transverse shear and normal stresses and strains might be present, but their contribution to the internal energy is considered negligible. Actually, Koiter (1960) intended with that kind of assumption to overcome the requirement of vanishing transverse normal strains and stresses, which is obviously contradictory.

The idea of Libai and Simmonds (1998) can easily be understood by looking at the constitutive law of the three-dimensional shell, Eq. (84). Suppose the vectors of kinematic and static variables would contain also the quadratic terms γ_{ij} and s^{ij}, respectively, whereas the corresponding terms in the material matrix are zero. Then, the additional kinematic variables γ_{ij} do not contribute to the potential energy, and consequently, the actual order of the theory is not influenced. This, in turn, means that the order of approximation is driven by the material law alone, regardless of the underlying kinematic assumptions.

7. Conclusions

In the present paper, a physical or mechanical, rather than a mathematical approach to the understanding of a higher order shell formulation has been provided. It has been discussed how higher order kinematic and static variables can be interpreted from an engineering point of view, and how they influence the accuracy of the shell model. It turned out, that the conventional association of the static variables with 'stress resultant' forces and moments is not possible for some of the higher order static variables.

Obviously, the 'step back into three dimensions' in shell analysis is realized predominantly by introducing a thickness stretch of the shell as independent variable. However, care has to be taken while choosing the stress and strain terms to be involved in order to avoid artificial stiffening effects. Here, it could be demonstrated that the format of the constitutive tensor, obtained by pre-integration of the three-dimensional material law across the thickness, plays a crucial role. It is stated, that it should always contain the full set of three-dimensional stresses and strains for each order in the thickness coordinate. For a theory including membrane and bending effects, like the 7-parameter theory discussed herein, this is the order one.

The study confirms that the 7-parameter shell formulation, initially proposed by Büchter and Ramm (1992a), and adopted later on by others (see e.g. Betsch et al., 1996; Eberlein and Wriggers, 1997) can be regarded as the lowest possible shear deformable shell model, which is able to handle arbitrary three-dimensional constitutive laws.

For an efficient numerical treatment of the shell equations with the finite element method, and numerical examples, including hyperelasticity and elasto-plasticity for large strains the reader is referred to Büchter et al. (1994) as well as Bischoff and Ramm (1997).

Acknowledgements

The present study is supported by grants of the German National Science Foundation (DFG) under project Ra218/11-3. This support is gratefully acknowledged. The authors would like to express their sincere thanks for the extremely helpful and constructive comments of the reviewers.

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